Linear Algebra
M. Math - I
End-Semestral Exam
2011-2012

Time: 3hrs
Max score: 100

Answer question **1** and any **five** from the rest.

(1) State true or false. Give reasons.
(a) If R is a non-zero commutative ring with 1 such that every ideal in R is free as an R-module, then R is a PID.
(b) The only simple modules over a PID R are of the form $R/(p)$ for some prime p in R.
(c) The ideal in $\mathbb{Z}[x]$ generated by 2 and x is a direct sum of cyclic $\mathbb{Z}[x]$-modules.
(d) 3×3 nilpotent matrices A and B over a field F are similar iff they have the same characteristic polynomial.
(e) The exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where π is the canonical projection, splits. (5+5+5+5+5)

(2) (i) State spectral theorem for normal matrices.
(ii) Let A be a normal matrix. Prove that A is hermitian if and only if all eigenvalues of A are real, and that A is unitary if and only if every eigenvalue has absolute value 1. (3+12)

(3) (i) Let I be a nilpotent ideal in a commutative ring R, let M and N be R-modules and let $\phi : M \longrightarrow N$ be an R-module homomorphism. Show that if the induced map $\bar{\phi} : M/IM \longrightarrow N/IN$ is surjective, then ϕ is surjective.
(ii) Let R be a ring and P be an R-module. Show that P is a direct summand of a free R-module (i.e. a projective module) iff every exact sequence

$$0 \longrightarrow M' \longrightarrow M'' \longrightarrow P \longrightarrow 0$$

of R-modules splits. (6+9)

(4) Let R be a PID and p a prime in R. Let F be the field $R/(p)$.
(i) Let M be a finitely generated free R module, say $M \approx R^r$. Show that M/pM is a vector space over F of dimension r.
(ii) Let $N = R/(a)$ where a is a nonzero element of R. How would you describe N/pN? (Consider cases where p divides a in R, or otherwise.)
(iii) Hence conclude that if $Q = R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k)$ where each a_i is divisible by p, then $Q/pQ \cong F^k$. 5+6+4

(5) (i) State the fundamental theorem for finitely generated modules over a P.I.D., in its invariant factor form.
(ii) If $p(x)$ and $m(x)$ are the characteristic and minimal polynomials of a $n \times n$ matrix A, and if $a_1(x), a_2(x), \ldots, a_n(x)$ are the invariant factors of A, where a_i / a_{i+1}, $i = 1, \ldots, n-1$, show that $p(x) = a_1(x)a_2(x)\cdots a_n(x)$ and $m(x) = a_n(x)$. (3+12)

(6) (i) Show that two square matrices of the same order are symmetric iff they have the same rational canonical form.
(ii) Find all similarity classes of 3×3 matrices A over the field of rationals $\mathbb{Q}$ satisfying $A^6 = I$. (8+7)

(7) (i) Show that any complex square matrix is similar to its transpose.
(ii) Find the invariant factors, elementary divisors, characteristic and minimal polynomials and the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 3 & 1 & -3 \\ -7 & -2 & 9 \\ -2 & -1 & 4 \end{bmatrix}.$$

(7+8)